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MOTIVES FOR THE CULTIVATION OF MATHEMATICS

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THE fundamental motives for its cultivation mathematics shares with the other sciences; for they and it are equally creations of the mind and derive their characteristic qualities from the mind, subject to varying color due to the differing materials. Motives which are special to the science of mathematics, if they exist, are less fundamental and are personal rather than general in their nature. In the light of the subject matter of mathematics the basic motives may exhibit a different color or a different mutual relation from that to be observed in the light of another particular discipline; and this variation may be worthy of special consideration. In a particular age or country the general motives may operate in unbalanced proportion so that an analysis of the situation may reveal improper tendencies which demand correction. It is for the purpose of ascertaining what changes, if any, are desirable in the distribution of motive as it now operates in the cultivation of mathematics in America that the considerations of this essay are presented.

It may be possible to classify motives into logically distinct groups so that there are no omissions and no overlappings; but it serves better our purpose to divide them into classes not mutually exclusive, classes among which there are vital connections analogous to those among the various parts of a living organism. It appears also that this is the better way to exhibit the science in its true aspect as a thing of life, growing under the action of ever-varying forces and impulses.

Probably the most fundamental impulse for the advancement of knowledge is that which grows out of the pursuit of truth for truth's sake. We are fundamentally so constituted that we delight in knowing for the sake of knowing. It is hard to describe this motive, or even to conceive it, in other than a vague way. It is our most abstract and our most general motive. It actuates most powerfully our choicest spirits, moving them sometimes with a fervor akin to that of religion. A marvelous curiosity to know creates a longing which can be satisfied only by knowledge. It projects itself into the unknown and leads the researcher in ways yet untrodden to a goal which

can not be foreseen. At the outer boundary line of knowledge, faint glimmerings may be detected in the darkness of ignorance beyond. What beckons us forth we do not know. Whether it can bring us any good we have no means of foretelling. It may lead us to a tragical something which will make it necessary for us, in much pain, to cast away some of our most cherished prejudices. But, whatever lies beyond in that which is concealed from our present vision,

> We work with this assurance clear,
> To cover up a truth for fear
>   Can never be the wisest way;
> By every power of thoughtful mind
> We strive a proper means to find
>   To bring it to the light of day.

Delight in the beauty of truth is a central incentive to its study and creation; and this operates with a unique and peculiar power in the cultivation of mathematics. In some respects its beauties are peculiar to itself and require the trained mind to perceive them, just as the deeper beauties of music (for instance) are perceived only by the practised ear of the musician. A much larger proportion of the excellencies of mathematical truth can be enjoyed by cultivated people in general than is usually supposed; but we still lack the exposition suitable to make this manifest. Both for the individual mathematician and for mankind at large a second fundamental motive for the cultivation of mathematics is that which grows out of the pursuit of high esthetic interests.

The leading characteristic of man is the power to think. There is nothing of higher esthetic interest than to determine whether we can think consistently. This fundamental question can be answered in the affirmative only by exhibiting the result of consistent thinking. The existence of mathematics gives the spirit of man leave to believe in itself, since here admittedly is a body of consistent thought maintaining itself for generations and even for millenniums.

But man is not all spirit. We can not live by intellectual delight alone. We have to get around in a world which has trees and stones and mountains and rivers, and shifting currents of force, and even living things which dispute with us the possession of the earth or are used by us for food or beasts of burden. Many of these opposing forces are physically far stronger than we. If we are to control them it must be through a superior knowledge of them and of our common surroundings. This knowledge is necessary to our welfare. Thus as a third fundamental motive for the cultivation of mathematics we have that

which grows out of the pursuit of means for interpreting and understanding our environment.

It is obvious that a single piece of mathematical work may be undertaken from considerations arising at once from two or from all of these three fundamental motives. There may be involved simultaneously the pursuit of means for interpreting and understanding our environment, the pursuit of high esthetic interests, and the pursuit of truth for truth's sake. But it seems that there is no other motive of fundamental importance which is not included in these, either separately or conjointly. That which finds its place here least naturally, perhaps, is the play motive which has certainly operated with considerable force among a few persons who have cultivated mathematics in the spirit of amateurs. But so far as a thing of this sort has been a fundamental motive in the general development of mathematics it may be associated with the delight arising from esthetic considerations or from the desire to know merely for the sake of knowing.

Even such an ideal motive as that arising out of the pursuit of truth for truth's sake has in it elements of danger, owing partly to the lack of clear definition of what is involved in it. It is easy to give it glibly as one's fundamental motive and so conceal from oneself a lack of thought on the matter or the lack of deep reality or genuine sincerity in one's motives. On account of its vagueness it may give rise to a sort of mysticism which does not consort well with scientific ideals. It runs the risk of becoming a fetish, an object of excessive devotion, and of drawing the veil over necessary distinctions in the values of truths. A catalogue of all the truths in the universe would probably be useless to limited beings like ourselves. The totality is so vast that we can not comprehend it or get about among its parts. We need some means of ascertaining what truths signify something for us, as we should otherwise be lost in the maze of all that is true and be unable to extract what is of value to us. In the pursuit of truth merely for truth's sake there is danger of giving attention alike to what properly concerns us and to what is without distinct relation to any of our needs. We require other motives to operate in the way of helping to direct our activities.

The demand for simplicity and elegance in the pursuit of esthetic interests in science may tend to render labor effeminate. Mathematicians must attack difficult problems with the zest of red-blooded vitality. They must not be repelled by complexities and inelegancies. The problem which is confronted in the search for truth must be solved, even though it be by tedious

means and with results hard to understand. To be sure, "a thing of beauty is a joy for ever;" but nothing is a thing of beauty which is merely so. That beauty is not permanent which is its own entire excuse for being. In scientific truth there is a need of sincerity and high purpose back of the creation of anything of beauty. To pursue truth merely and solely for the sake of esthetic delight tends to induce in one an admiration for the tinsels of knowledge and a joy in its more superficial elements. The greatest delight in the beauty of truth flows from its unfolding as an incident to the creation of values of profound import to mankind.

Gross utilitarianism is the obvious danger which arises from the pursuit of means for interpreting and understanding our environment. If we start out to create truth for the sake of its applications we take a one-sided and narrow view of it. By circumscribing our vision we fail to see the connection of related parts and our progress is soon brought to an end. Nature does not yield her secrets to him who seeks them to supply his immediate grosser needs; she rewards only a more idealistic purpose.

Worse than any of these dangers incident to an improper emphasis on the general motives is that arising in the case of an individual or a nation from false or selfish motives. From the ore of thought important truth can usually be extracted only under the heat of a glowing zeal, when the mind is surcharged with that determination which arises from strongly motivated activity. Work which proceeds not at white heat is coldly done and possesses not the fire of vitality. The mind can be brought into this fit attitude and activity only by means which are in accordance with its fundamental ways of working. A motive growing out of the desire for selfish values inspires no such state. Only minor results may be achieved under such a spur to activity. The young researcher who looks forward to a career of useful discovery should regard the cultivation of high motives in his own mind as one of his primary and most important duties. If he is not already moved by the higher considerations, there is little hope for him; if these now operate powerfully in his thought, let him seek means to develop them more fully, let him meditate upon the things of higher importance and more profound value so that these shall ever renew and build up in him ideals of the nobler sort. One can not successfully woo the science of mathematics (or any other science) except under the inspiration of high motives. She refuses to consort with sordid aims. She can be happy only with him of high ideals who cherishes her nobler qualities; and only to him will she yield her increase for the blessing of mankind.

Over against the dangers arising from an unbalanced emphasis on the fundamental motives are the peculiar advantages due to those activities which are inspired primarily by each of them separately. In pursuing truth for truth's sake we solve hard questions by the best methods we may; but we solve them, or else we keep them before us as an ever-present incentive to the creation of new methods of conquest and power. Difficulty never turns us aside, except temporarily while we seek new means of progress or investigate adjacent fields. The absence of apparent esthetic satisfactions is no bar. If we can not find the truth which delights us with its elegance and beauty we will ascertain the best possible. We shall brood over its incompleteness until we find a way of bringing it to perfection. If the matter connects with things which appear to us vital we shall pursue it to the end regardless of practical utility of any sort. An inner spring, a necessity of our being, impels us in this direction. We are fortified in the desire to follow up our inclination here by the observation that our predecessors in laboring under a like impulse have often found results necessary to the realization of other desired ends. The unknown is too mysterious to be charted in advance. For the best means of penetrating it we must trust largely to our blind instincts, modified perhaps by past experience but still maintaining their central characteristics. In this way we not only acquire new truth, but we also develop new methods of discovery, the most elusive thing in scientific progress. A method discovered in one field under the fire of a blazing zeal enables us to surmount elsewhere other difficulties of more immediate concern, perhaps, but lacking an element which brings us in the consideration of them to the highest state of concentration and creative activity.

In pursuing esthetic satisfactions we create a beautiful theory for the sake of our delight in it, as in the case of the theory of numbers or of abstract groups. Working in such fields with the simpler elements of mathematical thought we make progress of a sort not at first possible with the more complex materials. We bring the theory to a higher state of perfection; there are fewer lacunæ; the connections of the various parts are exhibited with clarity; we have a sense of having seen to the root of the matter and having understood it in its basic characteristics. The theory thus developed becomes an ideal in the light of which we get a new conception of what should be attained in other fields where the labor and the difficulty are greater. Results in one field of mathematics may thus become of great value in a totally different range of mathematical ideas or even in other disciplines altogether. Moreover, when such

progress is attained we often find that the tools employed in bringing it about are sufficient for dealing with more difficult matters, so that the one completed theory furnishes us not only the ideal, but also the means for further valuable progress.

A characteristic delight in mathematical truth is that which arises from economy of thought realized through the creation of general theories. When we develop the consequences of a set of broad hypotheses we find that our results, which are attained by a single effort, have applications at once in many directions. Thus we see the common elements of diverse matters and are able to contemplate them as parts of a single general theory pleasing for its elegance and comprehensiveness.

Whether we like it or not, the evolution of humanity is a part of the cosmic process. Investigation shows that it has been so in the past. All the means by which we explain development point to the conclusion that it will remain so in the future. Since we are a part of the cosmic process, our greatest good comes with our best understanding of it. Our direct and unaided intuition does not lead us far toward comprehending the complexity of our environment. We have not the power to see directly into the explanation of things or even to devise representations of phenomena. We must seek means to assist our weakness in overcoming the difficulties of understanding. We can afford to omit none which yields, or which promises to yield, useful assistance.

Mathematics has shown itself a valuable tool in the interpretation of phenomena. It has been successful to a marked degree. It is marvelous what sorts of things come within its scope and what connections it exhibits among them, as, for instance, in celestial mechanics, rational mechanics, kinetic theory of matter, and the theory of electricity and magnetism, to mention only a few. Through the help of mathematics we gain an increased control over nature, to our comfort and perhaps to our happiness. Through this we are able to supply our bodily wants more readily and therefore have greater freedom for meditation on the deeper things of existence—those things which we have not yet been able to bring under the domain of exact science, however vital they are to our general and to our individual development.

This contact with nature gives mathematics itself a fresh strength and a changed direction. Excursions into the domain of applications enrich the science with new conceptions, new problems and new methods. This is borne out by the history of the past, both the remote and the more recent. It is especially noticeable in the creation of some of the most important

disciplines and in the activity of some of the most renowned mathematicians.

It is instructive to consider the distribution of motive in the work of certain of the greatest mathematicians. We select a few from different ages and fields of mathematical activity and of varying temperament. The list of course might be greatly extended; in fact, this is done to some extent in our later consideration of certain specific topics. Those here chosen may perhaps be taken as representative of the class of mathematicians of leading importance.

We begin with Euclid. To what extent he was an original investigator is unknown; but he must have made important contributions, since otherwise his *Elements* would not so quickly have supplanted the work of his predecessors. He gave his writings a good form from the point of view of logical connection and also of pedagogical excellence. It is clear that he took delight in the beauty both of the content and of the form of his work and that he developed it primarily from the love of truth for its own sake. But his geometric postulates are what he believed to be the obvious properties of space, either of experience or of contemplation; and this part of his work may therefore be looked upon as a contribution to the study of that space in which all phenomena have their being. Still it is certain that our three general classes of motives did not operate in balanced proportion in actuating the work of Euclid, at least if we agree that the normal situation is that in which they should receive approximately equal emphasis.

Archimedes was probably the greatest mathematician of antiquity. He was inspired primarily by the love of pure science, rejoicing in the truth because it is the truth and feeling a certain contempt for the applications of truth in the way of supplying our grosser needs; and yet he was a great practical inventor and had a wide range of knowledge of phenomena, and frequently gained new strength by his contact with nature. He founded the theory of hydrostatics and contributed effectively to the initial development of mechanics and of astronomy, so that he is to be reckoned as an important figure in the history of applied mathematics even though it is true that his leading title to fame comes of his work in pure mathematics. In him we have another instance of contributions of high value associated with an unbalanced emphasis on the fundamental motives.

Fermat seems to have cultivated certain parts of mathematics for the pure love of their beauty. Probably he was hardly conscious of motive at all, since his activity was so nearly

spontaneous. He made important contributions and inspired great advances, primarily in the way of new impulses to certain isolated studies.

Newton was undoubtedly moved primarily by a desire to understand and interpret natural phenomena. To this motive therefore we owe his invention of the differential and integral calculus (shared with Leibniz), the founding of celestial mechanics and rational mechanics in general, and the consequent development of applied mathematics in many fields of science.

A very few individual men have stood out among their contemporaries as admittedly the greatest mathematicians of their respective ages and have had the good fortune to have this verdict sustained by later generations. Euler was one of these. His prodigious activity and the penetrating character of his ideas have been alike the admiration and the inspiration of his contemporaries and successors. He has touched almost every department of mathematical science and most modern subjects in mathematics (both pure and applied) are affected by one or more streams of influence from his genius. He was actuated by all three of our general motives. From his memoirs one may select typical cases of work actuated primarily by any one of our three classes of motives or any combination of them. Taken as a whole his work holds a just and balanced proportion among them. No man before him seems to have given so nearly equal emphasis to each of the three fundamental motives, and no one has ever maintained a more vital and vigorous enthusiasm enduring over so long a career of investigation. It is significant that this balance of emphasis was coupled with discoveries of the greatest range and magnitude and influence and importance.

The case of Gauss affords another instance where the three motives worked in proper proportion, and also another instance of one holding a place of preeminent importance and influence. He was inspired by the beauty of pure truth as exhibited, for instance, in the theory of numbers; he sought a deep and penetrating understanding of things for its own sake, as in his meditations on non-Euclidean geometry; he devoted much attention to the interpretation of natural phenomena, as in his study of electricity and magnetism. The range of his influence on the further development of mathematics has been as great as the variety of motive inspiring his work. He, as well as Euler, teaches us the value of balanced emphasis in motives, at least for those who are prime movers in the development of modern mathematical science.

Poincaré is the latest example of one to stand out definitely and admittedly as the greatest mathematician of his time. He

was actuated by all three motives in balanced proportion. He made fundamental contributions in many fields both of pure and of applied mathematics. No one can look at his work without seeing how he rejoiced in the beauty of truth. He has left on record a statement of his profound delight in science as the means of seeing, of knowing, and he has emphasized the fact that after all it is knowledge and insight alone which count. In the introduction to his first note on Fuchsian functions he says: "The aim which I propose . . . is to ascertain whether there exist analytic functions analogous to elliptic functions and suitable for the integration of linear differential equations with algebraic coefficients." It is known that his interest in differential equations was largely affected by their use in applied mathematics, so much so in fact that he was depressed when certain recent physical theories seemed to imply that differential equations are not so fundamental to the understanding of phenomena as he had supposed. Moved by the most profound motives and of the widest variety, operating over an extremely wide range of material, employing ideas of the most penetrating character, and applying his results in many directions, Poincaré stands out as the leading creator of mathematical truth in the past half century and one with few equals in the history of mankind.

Poincaré exhibited also a tendency, more marked in him perhaps than in any other mathematician, to consider a range of ideas which should probably receive increased attention owing to the growing complexity of modern mathematics (and science in general), namely, the tendency to analyze the elements of our progress in the light of broad philosophical principles. What we believe concerning the nature, the meaning, and the value of the truth with which we are concerned has a profound effect upon the operation in us of the motives for its creation.

If the illustrative cases which we have adduced are to be taken as typical of the best work in mathematics—and we have tried to make them so—they would seem to teach, among other things, that with the growth of mathematics there is a growing necessity for a proper distribution of motive in the work of the individual thinker if it is to maintain a place of preeminent importance in the development of the science. In the case of a nation or a people the same thing appears to be true in general. Early in its history a science may develop in parts in a one-sided and unproportioned way. But when it attains to maturity and each new advance must rest on a large body of results previously derived it is of increasing importance that a balanced distribution of motive be maintained.

It is instructive also to examine the distribution of motive in some of the most important subjects of recent progress, following the development without reference to the individuals by whom it has been brought about. To me it appears that the greatest recent advances have been made in the domain of analysis—in territory either directly belonging to it or closely connected with it by association—and that in this field are likely to arise our most fruitful investigations in the near future. Here we have a wealth of outstanding problems of broad character and of far-reaching importance. If it is so in algebra or in geometry, I am not aware of what these problems are. For this reason I have chosen the following illustrative topics primarily from the field of analysis.

Nothing is more characteristic of the modern element of rigor in mathematics than the theory of point sets. Viewed in its relatively completed state, it seems to be well removed from all considerations pertaining to an interpretation or understanding of our environment. In some aspects it seems almost to be merely a set of logical exercises created for themselves. In every way it has the appearance of a body of truth developed for its own sake under the impulsion of a desire to see the inner meanings and beauties of things where the intuition is in a large measure helpless. But a study of the chain of causes which led to the development of this theory carries us back to Fourier's investigations in the distribution and flow of heat and the fundamental function-theoretic questions which were brought into prominence in the discussion of his work. Thus we find all three classes of motives operative here, though one of them appears in a concealed form which is brought to light only by an examination of the history of the subject.

The extraordinary activity manifested a few years ago in the rapid development of the theory of integral equations was brought about by the conjunction of all of our classes of motives. The theory is elegant, and the body of truth developed is pleasing in its character and in its relative independence as a unit together with the many connections between it and other disciplines, and it has numerous direct applications to the interpretation of physical phenomena.

Many influences have operated to compel mathematicians to enter upon a study of functions involving an infinite number of variables. Our most simple means of representing general classes of functions of a single variable bring us to consider at once an infinitude of elements. The power series representation of a function, for instance, exhibits it as depending upon the infinite number of coefficients in such a representation; and

it is natural to consider how its properties depend upon and vary with those of these coefficients. A like problem arises in connection with the Fourier expansion and with many others. When one is embarked upon the study of functions of an infinite number of variables he cannot avoid the extension of his geometric conceptions so as to involve an infinite number of dimensions. The two things go together and afford mutual illumination. It is pleasing to see properties first developed for the finite case carried over to the infinite case, and our conception of the beauty of the system of truth is greatly enhanced when we see it in all the reach of its validity holding at once for all finite cases and for the infinite case. Naturally there are properties which distinguish and separate the finite from the infinite, so that some of them may be thought of as characteristic of the one and some of the other. We understand each of them better by seeing their analogies and differences, and thus we penetrate into a more satisfying realization of the nature and significance of the truth developed.

It is easy for one who has not meditated upon this matter to suppose that these are merely intellectual exercises with no other value than what is incident to the intellectual delight in them. But this is far from the truth. In fact, the phenomena of our environment have pressed these things upon us for a long time. Physical considerations brought us against the problem of an infinite number of variables long before we had any mathematical methods suitable for dealing with it. Let us take the intuitionally simple case of the motion of a uniform flexible string of given length and weight. This can be specified only by an infinitude of variable quantities, or coordinates, each depending on the time. In fact, physical phenomena usually depend in this way on an infinite number of variables. Lagrange's generalized coordinates furnish us one of the best and most remarkable means of studying motion—and it is to this that we try to reduce our interpretation of all physical phenomena. The Lagrange coordinates are finite in number for the simpler cases, but are infinite in number in the more general situations of nature.

For the development of the theory of functions of an infinite number of variables, both in the past and in the future, we have therefore the strongest sort of motives growing out of the beauty of the theory, the love of truth for its own sake, and the desire to understand better the environment in which we live. For this reason we may be sure that workers will be attracted to this subject and that it will have a great development, notwithstanding its inherent difficulty. We shall find after all

that much of it is elegant and many of its complexities will disappear in the light of the leading results which we shall attain.

Other progress in the same direction and under like impulses has been realized recently in the development of a theory of functions of curves and spaces. Again it is the physical considerations which have forced the problem upon our attention. This time it is the mathematical physicist who has formulated the new problem and laid the foundations of the consequent theory. If he seeks to study the potential due to an electric current in a fine wire—to take a simple case—it is clear that he has to do with a quantity which depends upon the shape of the wire and is varied by changes in the relative position of its parts. It turns out that there is so much of novelty connected with this new type of functions that some of the fundamental notions, such as that of derivative, for instance, are to be defined in a way not at first obvious. We are thus forced to a fresh analysis of the basic ideas of function theory in the light of a new body of material in which they are to find their use. This will certainly lead to a deeper understanding of these ideas and a more comprehensive view of the body of truth in which they are significant. To an unusual degree our curiosity is piqued to know the lay of the ground here and the direction in which the subject will develop. Already we are assured of its value owing to its many connections; and the concourse of all fundamental motives in a marked degree assures us of workers for the field and consequent progress of far-reaching character.

It is desirable for us to consider also the distribution of motive at certain periods of great mathematical advances in the more remote past. If we choose typical instances we shall be able to get some conception of the change of emphasis in motive and shall be able to see how the earlier is related to the more recent as put in evidence by the instances which we have just examined.

So far as mathematics was developed at all among the Egyptians it is clear that it was done in a crude way and essentially for its immediate practical uses. Such a spirit could not release a penetrating study and the consequent insight. It remained for the Greeks to rise to the higher motives associated with the love of truth and beauty and to lay the broad foundations of the science in the spirit which animates it to the present day. Due either to revulsion from the short-sighted vision of the Egyptians or (more likely) to the temperament of the Greeks themselves, the latter too largely ignored mathematical science in its aspect of usefulness in understanding the environment, giving their attention almost entirely to other matters.

The new point of view brought with it wonderful advances, but involved also certain elements to stand in the way of a continued and unbroken development.

The discovery of the existence of irrational magnitudes marks a significant event in the intellectual history of mankind. It was a matter of grave concern in the philosophical system of the Pythagoreans and for a long time they kept to themselves the awful and astonishing secret. In pursuing our study of these quantities (even down to the present day) we have been actuated primarily by the love of truth and beauty. No actual measurements of objects can reveal the presence of these irrational quantities, though it is easy to satisfy ourselves logically of their ideal existence. Only a part of our general motives are operative in their study.

The establishment of a vital and close connection between algebra and geometry was, in the mind of Descartes, a part of his search for a universal mathematical science which was to be only the prelude of a universal science of an all-embracing character. It grew out of a love of truth for its own sake. In geometry we seem generally to have emphasized this motive. It was so with the ancient Greeks, with Descartes, with the rise of modern pure geometrical methods, and with much of the recent development of the geometrical sciences. It is due to the inherent nature of this field of thought.

In the rise and development of the infinitesimal calculus we have a different state of affairs. Here there is scope for the vital activity of all three classes of motives and they have ever been conjointly in evidence. This is due in part to the extreme breadth of reach of the fundamental ideas of the calculus and in part to its peculiar fitness for the interpretation of motion, the basic element in terms of which we seek to interpret natural phenomena. That which grows out of the calculus and is intimately related to it is the most characteristic portion of modern mathematics and is primarily that which gives to it its large measure of importance. It is significant that it is also just the portion of modern mathematics in which our three classes of motives operate in the most nearly balanced proportion.

In the theory of functions of a complex variable, which is essentially an outgrowth and extension of the infinitesimal calculus, we have a field of truth which is rich in extent, in beauty, and in the quality of furnishing a means to interpret natural phenomena. In its initial rise, in its main features of interest, in the introduction of new ideas into it, and in its widening ramifications, we find constantly that activity which grows out of an intimate blending of the three general classes of motives.

It is the choice part of the most typical field of mathematical activity and illustrates beautifully the proper union and emphasis of motives.

The foregoing analysis of the work of a few mathematicians and the distribution of motive in the development of certain topics brings to notice three facts which are significant in this study: in the more recent mathematical work we have a definite tendency toward a more nearly equal emphasis on the three general classes of motives than is to be found in the earlier stages; this tendency is most marked in the more characteristic portions of modern mathematics; and this is true particularly in the case of the greater recent developments, especially of those in connection with broad general conceptions such as are present (for instance) in the theory of functions of an infinite number of variables and of functions of curves and spaces. This doubtless reveals a growing necessity incident to the greater complexity of the problems and the larger body of known truth on which the new discoveries must rest.

In our brief survey we have noticed how the greatest mathematical workers and the most important fields of mathematical thought have had an intimate connection with the interpretation of natural phenomena. England and France and Germany are the countries in which the most important and preeminent mathematical progress has been maintained for a long period of time coming up to the present. They have had workers actuated by all three of our fundamental classes of motives and a fairly well-balanced emphasis has been maintained among them. The cooperation of each with the others seems to have been essential to the progress effected. So far as mathematical research is concerned the English have leaned strongly towards its practical aspects, so much so in fact that they have suffered somewhat in their contributions; but in the last years there has been a growing tendency among them to correct the evil. In Germany all motives have operated strongly. Owing to the fact that the German university system teaches not only knowledge, but also research, no other country can show so many individual workers nor such a tendency to congregate into schools. Naturally, this is associated with much attention to minor problems and the doing of a large proportion of the drudgery incident to scientific progress. Fruitful ideas do not arise there as spontaneously as in some other countries, but no other people have shown a greater genius for developing the detail in connection with a leading fundamental idea once introduced. Mathematical progress in France during the past century has been of a most pleasing sort. All motives have worked in beau-

tiful cooperation and spontaneity of effort has led to the creation of many fundamental concepts. Here we see the best balance among the various motives and at the same time the most steady stream of progress. Each generation has been effective in a marvelous degree and their labors have not only enriched their own discoveries, but have also fructified mathematical thought throughout the world. Their experience and success seem to counsel the holding of a just balance among the three fundamental classes of motives.

So far, in America, we have realized progress in the pursuit of mathematical truth for truth's sake and in the pursuit of high esthetic satisfactions; but we have hardly realized anything in mathematics from the pursuit of means for interpreting and understanding our environment. This is strange in view of the distinctly practical turn of our people as a whole and suggests the opinion that we probably have an unused reservoir of strength which might become effective in the progress of applied mathematics. It is not to be supposed that a people of a practical turn of mind can produce mathematicians interested in the pursuit of ideal truth and of esthetic satisfactions and yet not have a source of strength for the development of the more practical aspects of the science. Some of our studies up to the present are adjacent to the field of applications; others are far removed from it; but very little of all that our mathematicians have done lies in the direction of a better understanding of natural phenomena.

We can not expect to maintain healthy progress by an unproportioned and one-sided development such as has characterized our work to date. Fortunately there is now a rising interest in America in applied mathematics. This should be developed and be guided into the best channels so that the work ultimately shall become of vital and far-reaching importance. By the nature of their previous work some of our mathematicians are definitely excluded from a leading part in this new development. They have devoted themselves to those fields of investigation which are far removed from the sciences of natural phenomena and therefore cannot turn their knowledge and experience to use in this direction. It is no cause for regret that this is so. In order to maintain as a people a balanced emphasis on motives we need to have individuals in which each fundamental class separately is dominant. But there are those whose labors have already led them to the borderland where pure and applied mathematics have a common boundary line. Some of these at least may step over into the adjacent field. For them to do so would seem to afford us our best and readiest

means of correcting the patent defect in our mathematical progress. We have already a young national tradition of high ideals in pure mathematics; let us as soon as possible realize the creation of a worthy tradition in applied mathematics.

A survey of the character of our contributions up to the present would probably suggest that our best point of entry is into the field of celestial mechanics, where in fact we have already done something, or into the theory of the partial differential equations of physics. Our previous labors in pure mathematics seem to have furnished us with tools suitable for use in either of these fields.

Every living science has two existences: one of them is objective, as in the body of scientific literature; the other is subjective, as in the minds of thoughtful persons and students of the particular science. The first is like a material body; the second is like the spirit. The first is enduring, like a stone or a mountain; the second is like a living organism, delicate in structure and highly susceptible to environment. Both of these existences are necessary to the progress of a science; indeed, necessary to its continued existence. The primary business of the researcher is to afford science its objective existence in the body of scientific literature. It is the business of the teacher to see that science has the second type of existence. Modern mathematics is now the heritage of a select few. We ought to make its great cultural elements a common property of cultivated people. But now it is notorious that they are unashamedly ignorant of this science. Nor is the fault to be laid entirely or even primarily at their door. We teach too much the mechanical aspect of mathematical reckoning and emphasize too little the great basic and fundamental notions which give to the science its vitality.

In each generation the most important labor that can be done is that expended in the creation or discovery of new truth. But this fact must not prevent our realization that the heritage of the past is to be preserved intact and transmitted to the future, not dead as in books, but living as in the minds of men and women. Not only must the line of progress be unbroken but many collateral branches must run out in all directions into the body of society, where scientific truth may bear fruit for the nourishment of mankind. A few persons will not suffice in this work of disseminating truth; many must be provided, if the truth is to be vital in the lives of our people generally.

As far as possible each individual should give his life to labors in which his spirit delights. This is necessary to the higher sort of intellectual achievement as well as to happiness.

Unless one rejoices in the realization of the general motives already treated there is no compelling reason why he should devote himself to mathematics; at most he can be only a weary plodder, whatever station he takes. Again, unless the matter of mathematical truth in its broader aspects exercises a deep influence over his meditations his labor in that field can not be particularly useful. To devote himself to it would be to waste his life, to spend in fruitless endeavor the energy which might be valuable if employed in more congenial pursuits. But the enthusiastic man or woman of merely moderate training has a place of importance in making mathematics live in the lives of the young and thence in the lives of the older. This is a labor in which one may take delight and through which one may project an influence into the future that shall work for permanent good.

A few can go further and render a more vital sort of service. A mere modicum of creative work of fair quality is of great value in the way of increasing the vitality of the teacher. The fact that he has created will illuminate the subject for him and give a different color to his teaching. His emphasis becomes better proportioned to the relative importance of the various topics and he is able to light his subject with the glow of personal fire and touch.

A significant measure of research is particularly important in the case of the teacher who prepares text-books for use in the more advanced undergraduate courses and for the first year of graduate work. It is unfortunate when embryo mathematicians are led over the basic portions of fundamental disciplines through the guidance of a dead exposition such as will usually emanate from one who has not himself advanced the subject in question or made use of its main ideas or results in some researches of his own. In America we stand now in danger of an increased number of these expositions lacking vitality. This is due to the concurrence of two or three causes. We have now a keen realization of the fact that there is among us a dearth of elementary expositions of the fundamental subjects. Our research men are engaged so zealously in their own investigations that they have usually not taken the time to prepare expository treatments; but this is an evil which seems to be in a state of progressive correction. In many of our institutions—especially among those which have lately aspired to positions of the higher importance—there is a peculiar (and sometimes an even dangerous) pressure upon the members of the faculty to produce publications of a professional sort. Some of the persons involved have not the training or the aptitude for research.

There is a danger that an increasing number of these will seek an outlet for their activity in the production of text-books. Already we have too much of this, so far as the more elementary texts are concerned. Let us hope that we shall be able to prevent the spread of the evil to the field next higher. One satisfactory way of offsetting this danger would be afforded by a greater willingness on the part of our research men to prepare introductory treatments, each one in his own field.

It is only when one is able to devote a large share of his energy to research and is successful in the creation or discovery of important new truth that he may rejoice in the fullest glow of delight through a realization in himself of all three general classes of motives. However important the work of instruction may be in itself and however far-reaching its stream of influence flowing in hidden ways in the minds of men and women, it can not be placed in the same category with that creative work which guides instructor and student alike and teaches generations what to think. He who discovers a fact or makes known a new law of nature or adds a novel beauty to truth in any way makes every one of us his debtor. How beautiful upon the highway are the feet of him who comes bringing in his hands the gift of a new truth to mankind!

No voice I raise to magnify the man
    That forms again the thought whose living fire
A palpitating spirit nobler than
    His own first warmed; his toils no zeal inspire.
    Of him, more worthy far, with joyful lyre
I sing, whose precious bit of novel truth,
    Revealed through labors long or hardships dire,
He lays before the feet of man. In sooth
Through him the aged world retains undying youth.

No tragic note inspires this quiet song.
    Abiding joy in truth is its deep spring,
And such delight as comes in labor long
    To him who learns by thinking everything
    His powers can clear of its dark covering.
When we the mind itself for truth entreat
    Or seek the hidden laws of happening,
By worthy labors make we conquests sweet—
Conquests of one without another's pained defeat.

No danger gives the keen Researcher zest;
    His languor no excitement can forestall;
No pain stirs deep emotion in his breast.
    He finds delight in truth or large or small,
    In those whom interest or pleasure call
Our ambient environment to scan,
    In fresh control of nature's forces all

Through better understanding of her plan,
In glimpses new into the growing soul of man.

How grew the keen Researcher? How? He grew
In common ways of life to manhood's state
And felt with gladness strength to strength accrue
As each new-found experience lent its weight
To serve the old or novel thought create.
The constant play of THINGS upon his soul
Remade in him whatever was innate
And moved him with the sense of deep control
Wrought out in him by forces centering from the WHOLE.

Into his inner thought these forces come;
They bring the flavor of the world outside;
And in his inmost heart, no longer dumb,
They speak in gracious accents to confide
Their meaning, blessing him as yet untried.
Not long with him alone a truth remains;
But, redirected, moves from this new guide,
And as it leaves a greater strength attains,
But holds new character the chief of all its gains.

Thus the Researcher early learned the truth
Which entered more and more into his thought
As manhood's state in time succeeded youth
And years grew longer decades while he brought
All power of mind to bear on what he sought:
Whatever comes beneath the busy hand
Of man, on which with power it has wrought,
Goes forth transformed in newness to expand
To what it was not, gladly doing his command.

The life of man is wrought of various parts
Which hold together in complexity;
No hand is skilled in divers proper arts
To draw the line of truth upon the sea
Where waves of change surge forth in full degree.
A separated part must one procure
Where change is not too vast for him to see
Its secret, if he makes a conquest sure
Which through the ages of the future shall endure.

'Tis this the watchful, keen Researcher sees:
It guides him to a bounded field of thought;
It teaches him the need of new degrees
Of power by which his conquests may be brought
To bear on widest realms with blessing fraught.
A zeal arises in his inmost heart,
A consecration by deep purpose wrought
Absorbs his strength, his life he sets apart
To bless mankind by streams of truth which from him start.